

Aging dynamics and the topology of inhomogenous networks

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We study phase ordering on networks and we establish a relation between the exponent a_χ of the aging part of the integrated autoresponse function χ_{ag} and the topology of the underlying structures. We show that $a_\chi > 0$ in full generality on networks which are above the lower critical dimension d_L , i.e. where the corresponding statistical model has a phase transition at finite temperature. For discrete symmetry models on finite ramified structures with $T_c = 0$, which are at the lower critical dimension d_L , we show that a_χ is expected to vanish. We provide numerical results for the physically interesting case of the $2-d$ percolation cluster at or above the percolation threshold, i.e. at or above d_L , and for other networks, showing that the value of a_χ changes according to our hypothesis. For $O(N)$ models we find that the same picture holds in the large- N limit and that a_χ only depends on the spectral dimension of the network.

In recent years, networks and graphs have been successfully applied to very different disciplines [1]. This extreme flexibility is not surprising, since they are the most general way of representing a set of elements connected pairwise by some kind of relation. In social sciences, computing, psychology and economy, graphs represent communication systems, social relationships, biological interactions and statistical models of algorithms. In physics, chemistry and biology, they are extensively used as models for real complex and inhomogeneous structures, such as disordered materials, percolation clusters, glasses, polymers, biomolecules. Networks representing real physical structures are constrained to be embeddable in a finite dimensional space and to have bounded coordination number, forming an interesting class of structures called "physical graphs" [2].

Basic physical equilibrium properties of models defined on physical graphs, among which critical behaviors, depend crucially on the topological features of the network. In particular, the relation between large scale topology and critical properties is well understood on a simple class of network, i.e. regular lattices, where the existence of phase transitions, the value of critical exponents and many other equilibrium quantities depend on a topological parameter, the Euclidean dimension d of the lattice. As far as systems with continuous symmetry are concerned, such as $O(N)$ models, this general picture can be extended to networks. Their topology can be completely described by the adjacency matrix $A_{i,j}$, with $A_{i,j} = 1$ if i and j are connected by a link and $A_{i,j} = 0$ otherwise, but the relevant large scale topological information is encoded in the Laplacian matrix $\Lambda_{i,j} = \delta_{i,j} - z_i$, where $z_i = \sum_j A_{i,j}$ is the number of neighbours of site i . Indeed, the density $\rho(\lambda)$ of eigenvalues of Λ , $\rho(\lambda) \sim \lambda^{(d_s/2)-1}$ for $\lambda \rightarrow 0$ defines the "spectral

dimension" d_s of the graph. The spectral dimension univocally determines the existence of phase transitions [3] and controls critical behaviour [4], much in the same way as the Euclidean dimension d does on usual translation invariant lattices. In systems with a discrete symmetry, although a unique topological indicator, analogous to d_s , is not known, the fundamental role of topological and connectivity properties in determining equilibrium and critical behavior has been pointed out [5].

The picture is much less clear for out of equilibrium processes. Although it is known that dynamical processes are influenced by the topology of the network, a direct description of this relation is largely incomplete. This is the case for the phase ordering dynamics, occurring in systems gradually evolving from an homogeneous phase to a state with two or more coexisting ordered phases, through formation and growth of segregated domains [6]. Initially investigated in physics, phase ordering also occurs in other fields where the topology is of primary importance, as in biology, where it describes spreading of tissues [7], and in economic and social networks, where it has been used as a model for propagation of opinions and technologies [8]. A deeper understanding of the relation between the dynamical process and the topology of the network in these frameworks would be of great interest.

In the simpler case of regular lattices, the influence of the Euclidean dimension on dynamical exponents has been ascertained in phase ordering in spin systems [9]. The generalization of such models to networks is provided by the Hamiltonian $H\{\vec{\sigma}\} = \sum_{i,j} A_{i,j} \vec{\sigma}_i \cdot \vec{\sigma}_j$, where the indices i and j run over the N sites of the graph, and $\vec{\sigma}_i$ represents an N -dimensional unit vector defined at the site i . For $N = 1$, H describes the discrete Ising model and for $N > 1$ the $O(N)$ models, with continuous symmetry. Phase ordering dynamics occurs when the system,

initially at thermal equilibrium at high temperature, is quenched to a temperature where the symmetry is broken and more phases coexist. In this scenario, an interesting physical quantity is the integrated autoresponse function $\chi(t, s)$ describing the effect of a small perturbation acting on the system from time s to t : $\chi(t, s) = \int_s^t R(t, t') dt'$, where $R(t, t') = N^{-1} \sum_i \delta(\vec{\sigma}_i(t) \cdot \vec{n}_i) / \delta|\vec{h}_i(t')| \vec{h}_i(t') = 0$ is the average linear response function associated to an impulsive magnetic field switched on in i at t' [10] ($\vec{n}_i = \vec{h}_i / |\vec{h}_i|$ and $\langle \dots \rangle$ means ensemble averages). Interestingly, $\chi(t, s)$, can be splitted into a stationary and an aging term [9] $\chi(t, s) = \chi_{st}(t-s) + \chi_{ag}(t, s)$, and the aging part features a scaling behaviour with a characteristic exponent a_χ

$$\chi_{ag}(t, s) = s^{-a_\chi} f(t/s). \quad (1)$$

In phase ordering on lattices [10]

$$f(x) \sim x^{-a_\chi} \quad \text{for } x \gg 1, \quad (2)$$

and a_χ is related to d by

$$a_\chi = \theta \frac{d - d_L}{2} \quad (3)$$

for $d < d_U$, while $a_\chi = \theta$ for $d \geq d_U$. Here d_L is the lower critical dimension, $d_U = 3$ or $d_U = 4$ for discrete and continuous symmetry models respectively, and θ is the exponent regulating the time decay of the topological defects density $\rho \propto t^{-\theta}$, which does not depend on d . By means of Eq. (3), a non-equilibrium exponent, a_χ , is related to the topology of the underlying lattice, through d . This relation implies that $a_\chi > 0$ when the system is above d_L , i.e. when a phase transition at finite temperature T_c occurs, $a_\chi = 0$ at d_L and $a_\chi < 0$ below d_L .

These results, holding in the simple case of regular lattices, suggest that non-equilibrium dynamics of statistical models could be related to important topological properties of networks. With the aim of testing and investigating this hypothesis, we study the response function exponent a_χ on generic physical networks, showing that i) $a_\chi > 0$ is expected in full generality on networks supporting phase transitions at finite T_c , i.e. above d_L ; ii) for continuous symmetry $O(\mathcal{N})$ models, in the soluble large- \mathcal{N} limit on graphs the same expression (3) is found, with d_s occurring in place of d . This fits with the aforementioned general picture that, for continuous symmetry models d_s is the topological indicator replacing d on graphs; iii) for discrete symmetry models, we provide an argument showing that $a_\chi = 0$ is expected on finitely ramified networks (FRNs)[5], which represents the most general class of structures without phase transitions at finite T_c . Although in this case a topological parameter playing the role of d_s is not known and, therefore, a generalization of Eq. (3) is not straightforward as for continuous symmetries, this result conforms with Eq. (3)

and provides, via the existence of phase transitions, a relation between the positivity of a_χ and the topological feature of the network. Our results are complemented by extensive numerical simulations of different discrete symmetry models on a representative set of graphs finding $a_\chi = 0$ or $a_\chi > 0$ depending on the absence/presence of phase transitions. In particular, we provide results for the physically interesting case of the $2 - d$ percolation cluster at or above the percolation threshold, i.e. at or above d_L , showing that the value of a_χ changes according to our hypothesis. Taking advantage from this, a_χ could be conveniently used to investigate the phase diagram of statistical models on general discrete structures.

Let us start showing that i) $a_\chi > 0$ is expected for models with a phase transition at finite T_c . Reparametrizing the two time dependence of $\chi(t, s)$ in terms of the autocorrelation function $C(t, s) = 1/N \sum_{i=1}^N \langle \vec{\sigma}_i(t) \cdot \vec{\sigma}_i(s) \rangle$ [11], one obtains the parametric form $\hat{\chi}(C, s)$, which is related to the structure of the equilibrium state through [12]

$$-T \lim_{s \rightarrow \infty} \frac{d^2 \hat{\chi}(C, s)}{dC^2} \Big|_{C=q} = P_{eq}(q) \quad (4)$$

where $P_{eq}(q) = Z^{-2} \sum_{\vec{\sigma}, \vec{\sigma}'} \exp[-\beta(H\{\vec{\sigma}\} + H\{\vec{\sigma}'\})] \delta[Q(\vec{\sigma}, \vec{\sigma}') - q]$ is the equilibrium probability distribution of the overlaps between two configurations $\{\vec{\sigma}\}$ and $\{\vec{\sigma}'\}$, $Q(\vec{\sigma}, \vec{\sigma}') = (1/N) \sum_i \vec{\sigma}_i \cdot \vec{\sigma}'_i$, Z being the partition function. General considerations can be drawn on the relation between χ_{ag} and the critical behavior of the corresponding statistical model. Let us consider a system with $T_c > 0$ quenched to $0 < T < T_c$. In this case the form of $P_{eq}(q)$ implies [13] that χ_{ag} must vanish asymptotically. Then, from Eq. (1), one has $a_\chi > 0$. Even if for quenches to $T = 0$ Eq. (4) does not give informations on χ_{ag} , since dynamical exponents are known [6] not to depend on T [6], $a_\chi > 0$ must be found in the whole interval $0 \leq T < T_c$. These considerations being completely general, they apply to networks as well, and $a_\chi > 0$ for quenches to $T < T_c$ in systems with finite T_c .

Let us now consider point ii). In the large- \mathcal{N} limit, the order parameter ϕ_i can be expanded in the base of the eigenvectors of the Laplacian operator and the dynamical evolution of the projection $\hat{\phi}_k$ of the order parameter on the k -th eigenvector of the Laplacian operator obeys [14]

$$\frac{\partial \hat{\phi}_k(t)}{\partial t} = -[\lambda_k + I(t)] \hat{\phi}_k + \hat{\eta}_k(t) \quad (5)$$

where λ_k is the k -th eigenvalue of Λ , $I(t) = 1/N \sum_i^N \langle \phi_i^2 \rangle - 1$ and $\hat{\eta}_k(t)$ is the thermal noise. Eq. (5) is formally linear and it can be solved in the large time domain. Proceeding as in [15] to compute $\chi_{ag}(t, s)$ for a quench below T_c and extracting the leading term, corresponding to the low eigenvalue behaviour of $\rho(\lambda)$, we

obtain the forms (1,2), with $a_\chi = (d_s - 2)/2$ for $d_s < 4$ while $a_\chi = 1$ for $d_s \geq 4$. This expression of a_χ is analogous to Eq. (3), with the Euclidean dimension replaced by d_s , $d_L = 2$, $d_U = 4$ and $\theta = 1$, as appropriate to vectorial models [6]. This also implies that when $d_s > d_L$, i.e. $T_c > 0$ [3], $a_\chi > 0$, consistently with our previous argument. Interestingly, most networks representing real physical structures have $d_s < 2$ [16] and, therefore, $a_\chi < 0$. This would imply a diverging response.

Let us now turn to discrete symmetry systems without phase transition, i.e. at d_L . In this case $T_c = 0$ and aging dynamics can only be observed at $T = 0$ where Eq. (4) does not provide a positivity constraint on a_χ . For instance, the exact solution of the kinetic Ising chain [17] gives $a_\chi = 0$ for $T \rightarrow 0$. In the following we introduce a topological argument showing that the same result $a_\chi = 0$ holds for coarsening systems on FRNs. On these structures, each arbitrary large part can be disconnected cutting a finite number of links, as those marked x or y in the representation of Fig. 1. For FRNs $T_c = 0$ [5], and even if the general condition for $T_c = 0$ for discrete symmetry models is not known, the available results indicate that large scale connectivity is the only relevant feature and that infinite ramification on scale invariant structures implies $T_c > 0$ [5].

Let us consider phase ordering on a FRN. We assume a no bulk flip rule (NBF), where spins in the bulk of ordered domains cannot flip. It has been shown [18] that this dynamics isolates the aging part of χ leaving other properties of the system unchanged. Switching on a random magnetic field with expectations $\bar{h}_i = 0$, $\bar{h}_i h_j = h^2 \delta_{ij}$ from time s onwards, χ_{ag} can be computed as [10, 13]

$$\chi(t, s) = \lim_{h \rightarrow 0} \frac{1}{Nh^2} \sum_i^N \overline{\langle \sigma_i(t) h_i \rangle}, \quad (6)$$

where $\sigma_i = \pm 1$ and $\overline{\cdots}$ denotes an average over the realizations of h_i . Let us focus on a domain of, say, up spins whose interface lies at time s on the link denoted x , and is subsequently found in y at time t . The domain's size grows from $L(s)$ to $L(t)$. The number of spins in the region B between x and y is $n_B \propto [L(t) - L(s)]^{d_c}$, where d_c is the connectivity dimension of the network [2]. We want to estimate the increase of the response $\chi_{ag}^{sing}(t, s)$ associated to the displacement of this single interface from x to y . Eq. (6) gives $\chi_{ag}^{sing}(t, s) = -(Nh^2)^{-1} \sum_y E_h P_h(y, t)$, where $P_h(y, t)$ is the probability to find the interface in y at time t and $E_h = -\sum_{i \in B} h_i \sigma_i = -\sum_{i \in B} h_i$ is the variation of magnetic energy due to its displacement from x to y . Now we make the assumption that the correction to the unperturbed probability $P_0(y, t)$ is in the form of a Boltzmann factor $P_h(y, t) = P_0(y, t) \exp(-\beta E_h) \simeq P_0(y, t)[1 - \beta E_h]$. Then $\chi_{ag}^{sing}(t, s) = -(Nh^2)^{-1} \sum_y E_h [1 - \beta E_h] P_0(y, t)$. Since the linear term in E_h vanishes by symmetry we find

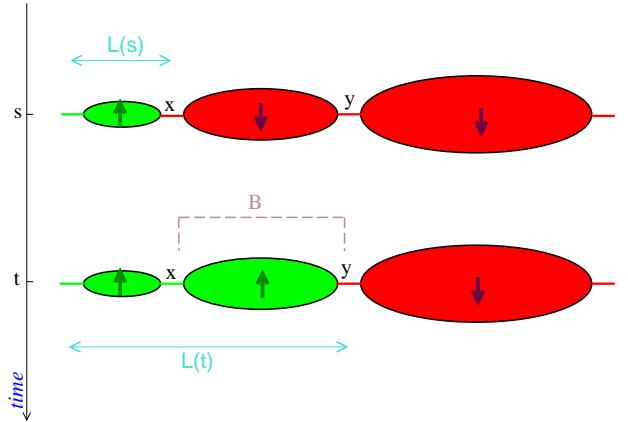


FIG. 1: Schematic representation of a FRN.

$\chi_{ag}^{sing}(t, s) = \beta(Nh^2)^{-1} \sum_y E_h^2 P_0(y, t) = \beta(Nh^2)^{-1} \overline{E_h^2}$. Recalling the expression of E_h and the expectations of h_i one has $\overline{E_h^2} \simeq h^2 n_B$. Then $\beta^{-1} \chi_{ag}^{sing}(t, s) \propto N^{-1} [L(t) - L(s)]^{d_c}$. The total aging response of the system at time t is obtained multiplying $\chi_{ag}^{sing}(t, s)$ by the number $n_i(t) = NL(t)^{-d_c}$ of interfaces present in the system. Then, for $t \gg s$ one has $\beta^{-1} \chi_{ag}(t, s) = n_i(t) \beta^{-1} \chi_{ag}^{sing}(t, s) \sim const$. Recalling Eq. (2), this implies $a_\chi = 0$. Putting together i) and iii) we conclude that $a_\chi > 0$ or $a_\chi = 0$ is expected in the phase ordering of discrete symmetry systems on networks with $T_c > 0$ or $T_c = 0$ respectively.

In order to check this general result, for the Ising model we have determined numerically a_χ in a representative class of structures. We have used the NBF to extract the aging part of two time functions. Notice that the NBF suppresses bulk fluctuations and hence rises the critical temperature [19], allowing one to avoid numerically demanding low temperature regions. The parametric plot of $\widehat{\chi}_{ag}(C_{ag}, s)$ versus C_{ag} , is well suited to discriminate between $a_\chi = 0$ and $a_\chi > 0$. In fact, since $C_{ag}(t, s) = g(t/s)$ [6], with g a monotonously decreasing function, from Eq. (1) one has j) collapse of $\widehat{\chi}_{ag}(C_{ag}, s)$ on a single master-curve, for $a_\chi = 0$ or jj) lowering of the curves increasing s , for $a_\chi > 0$.

Coarsening on a percolation cluster is suited to describe our results because the bond probability p distinguishes a situation with $T_c = 0$ at the percolation threshold $p = p_c$ [20], from those with $T_c > 0$ when $p > p_c$. Fig. 2 shows that indeed for the Ising model one has $a_\chi = 0$ at $p = p_c$ and $a_\chi > 0$ for $p > p_c$. Furthermore, we have performed analogous simulations on two FRNs with $T_c = 0$, the Sierpinski gasket and the T-fractal (Fig. 3), where $a_\chi = 0$ with great accuracy (Fig. 4). The situation is different for infinite ramification, with $T_c > 0$, e.g. on the Sierpinski carpet and the Toblerone lattice [21] $a_\chi > 0$ (Fig. 4). Similar results are found for different discrete symmetry models with non-conserved order parameter [6], such as the 3-state Potts model, confirming

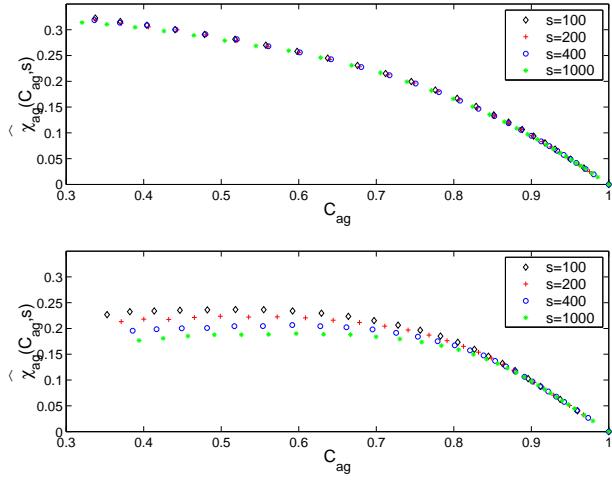


FIG. 2: $\langle \hat{\chi}_{ag}(C_{ag}, s) \rangle$ for the Ising model quenched to $T = 1.25$ on the site-percolation cluster at $p = p_c = 0.407$ (upper panel) and at $p = 0.2$ (lower panel). The cluster is built on a 1200^2 square lattice. Data are averaged over 100 realizations.

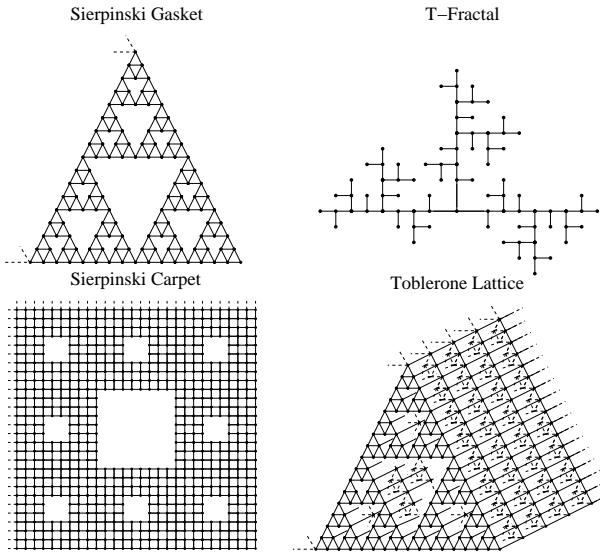


FIG. 3: The discrete networks considered in simulations: 265722 sites Sierpinski gasket, 531442 sites T-fractal, 1048576 sites Sierpinski carpet and 5078988 sites Toblerone lattice.

the generality of our result. Interestingly, in all our simulations, the value of a_χ has been found to be extremely stable and independent of the temperature at which the simulation was performed, even when other asymptotic quantities were strongly fluctuating. This suggests that a_χ should be related to a true universal characterization of the process.

In conclusion, we have shown that, in all the cases considered, statistical models on networks above d_L are characterized by a positive non-equilibrium exponent a_χ , while structures at d_L have $a_\chi = 0$. For continuous symmetry systems, our large- N calculation shows that, besides this, the whole dependence of a_χ on the network

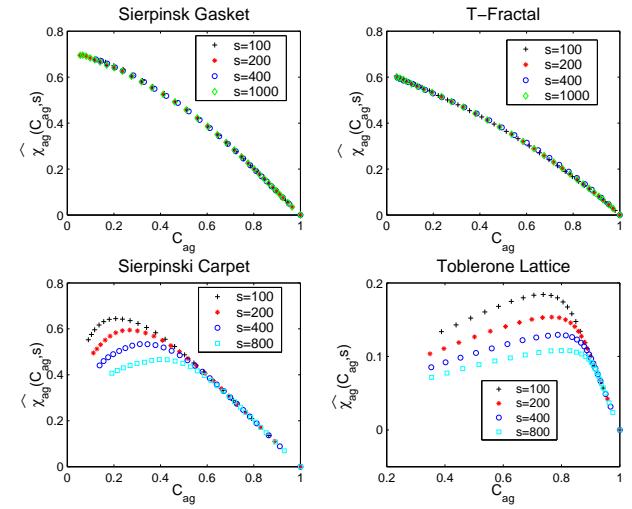


FIG. 4: $\langle \hat{\chi}_{ag}(C_{ag}, s) \rangle$ for the Ising model at $T = 3.0$. On FRNs $a_\chi \simeq 0$; for the Sierpinski Carpet and the Toblerone lattice $a_\chi \simeq 0.17$ and $a_\chi \simeq 0.25$.

topology can be expressed as for regular lattices, namely Eq. (3), with the spectral dimension d_s replacing the Euclidean dimension d . Our results provide an evidence for a relationship between non-equilibrium kinetics and large scale topology on general networks and suggest that the same topological features of the networks determine critical behavior and non-equilibrium exponents of phase ordering.

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